Pell Equation

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Introduction

One of the biggest mathematical biases is the inability to recognize spacial maths versus number line maths. This is usually expressed by using pi, squaring and square rooting as if we are talking about the number line. That fact that some number line functions return partially similar results to spacial plane functions confuses people.

The book - ***The Pell Equation*** *by* ***Edward Everett Whitford*** *was written in 1912 – college of the city of new York and submitted in partial fulfillment for a degree in the doctor of philosophy in the faculty of pure science Columbia University.*
https://archive.org/details/pellequation00whitgoog/page/n7/mode/2up . It is difficult to discover much about ***Edward Everett Whitford*** – most references only give his date of birth 1865 and that he was a math teacher and a member of the American Mathematical Society; there are records of his attendance at meetings. His book is well written and researched and contains many references to Authors and

materials in a variety of languages and who are not all common names now although in their time they were much more well known. "*references to over 300 Authors*". **Edward Everett Whitford** seems to have been able to navigate and understand Greek, Latin, French, English and German texts.

I can imagine him going to the **New York Public Library** as well as the universities – to do hours of research. It now has over 900,000 digitized items https://digitalcollections.nypl.org/. He wrote papers on the issues of teaching of mathematics to students.

Framework

Universe, Bounds, Constraints

Population

Universe

Questions

1. What is interesting about the equation?
2. What were Fermat, Euler, Brahmagupta and others thinking?

Initial Conditions

Universe, Notice, Observe, Bias

Self reference

Bounds and Constraints, Pythagoras

Disclaimer

These are my own thoughts and I have provided links to many resources. I have extracted some information from other people and tried to present it and attribute it –generally as a **fair use** – extract or link - **research** and **education** and **non–commercial** reference. I refer to group and individual's work and exploration – not as a personal reflection, criticism or evaluation – but recognition of the importance of their work for humanity in general. I try to name the sources of information used, extracted or explored while writing this paper. I have added my own **emphasis** and **highlights** to extracted text to highlight the points I am trying to explore.

Any objections can be referred to my website and I am not unwilling to remove references or material if there is a problem.

The bulk of the material accessed has generously been made publically available for research by many people – I appreciate their work and acting in the public interest.

Additional and supplementary information is presented in References.

Initial Thoughts

Many people dive into math equations without examining the question more thoroughly.

This Pell equation held some interest for many people but few seem to be able to explain why.

Fermat noticed that the bias towards geometry was overlooked by many mathematicians.

Initial Exploration

Edward Everett Whitford explains much in his book – the history of this equation is long and it has much to do with squares and square roots. This seems at odds with Fermat's view on *"pure arithmetic"* but maybe Fermat was just highlighting how he could easily switch between geometry and arithmetic.

Historically the sun, moon, stars dominated the sky – the upright stick become the sundial – the shadow drew lines – semi-circles, perpendiculars, right angles, triangulation and squares appeared in daily life. Shapes and numbers had special relationships which formed patterns.

Stones in circles, blocks , squares , drawings – dating back over human existence. Exploring patterns – leaving messages for those who followed. Different societies embraced geometry and numbers in different ways and for many years it was considered special knowledge – kept by special humans.

Sundials, ropes with 12 knots, measuring stones, sticks, etc – were special tools to help spacial maths.

Brahmagupta was no stranger to Astronomy and he was trying to understand numbers – so it was natural to have a geometric view – the Greeks had publicized maths and formulas – **Pythagoras** was taken for granted, **Euler** had discovered walks and bridges and was also thinking in nodes and edges.

Everyone was trying to reconcile everything – find the "universals" – the hidden truth.

https://en.wikipedia.org/wiki/Pell_number ***Creative Commons Attribution-ShareAlike 3.0*** *.."In mathematics, the **Pell numbers** are an infinite sequence of integers, known since ancient times, that comprise the denominators of the closest rational approximations to the **square root of 2**"*

https://en.wikipedia.org/wiki/Lucas_sequence ***Creative Commons Attribution-ShareAlike 3.0****..."Famous examples of Lucas sequences include the Fibonacci numbers, Mersenne numbers, Pell numbers, Lucas numbers, Jacobsthal numbers, and a superset of Fermat numbers. Lucas sequences are named after the French mathematician **Édouard Lucas**."*

Initial self-reference and recursion.

The Pell equation is a simple statement with a lot of meaning behind it. It is worth exploring the context a bit further. My ideas – may be similar to some other people – but I think there are a few differences or maybe assumptions people have made which I want to explore a bit. As I re-read parts of Plato's Republic – especially his section about the relationships of numbers – I become more aware of what he knew and had seen about the universe.

I am generally reading things now with an attempt discover what was assumed or missed rather than just what is stated.

Some Problems Stated

The equation $\mathbf{x^2 - ny^2 = 1}$ is a version of the **Pell equation**.

The most common related equation is $x^2 + y^2 = z^2$ which expresses Pythagoras and the relationship to **Right Angles** and **Squares** and **Diagonals**. We also get in **implied order** to the statement if x = 3 and y = 4 **then** z = 5. We also get implied **initial self reference** because we have no choice but to consider the **Square Root** as a process as well. It is assumed in the statement because it is the technique we need to use to examine variables (x,y,z) and consider from a possible infinity of options – which ones satisfy the equation - "solving" a hypothetical relationship.

The assumptions are not all clearly stated.

Spacial awareness is not unique to humans and we see it in all life forms. We see the most primitive creatures reaching out and testing a surface – searching – looking for a good spot to take the next step.

So this **direction** thing **is important**.

1) How did we get here?
2) Where can I go?

This sums (get it) up the issue with the Pell Equation and Pythagoras. One is a kind of forward question and the other backwards. I always have trouble with integration versus differentiation – this why conceptually advanced mathematicians – especially looking at prime numbers - often do logloglogloglog dance routines (because many other humans cannot).

In a Pythagorean sense the Pell Equation has done some clever things. It has switched the square root of 2 (the hypotenuse) to be equal to a diffrent unit and we see the square function (and implied root) but we also see a minus sign. This minus sign **helps us reorient our minds** to a "how did we get here" type mindset. One side of the triangle has been standardized to equal 1.

Another clue is the use of variables. The use of x y and n means we are looking at relationships. If we are looking at relationships we are looking at things spacially – looking for distances and areas. Fermat might think that arithmetic can solve the equation better than geometry – but it is still a geometric statement. The other clue to it being about geometry is the use of squared – a thing multiplied by itself. This **initial self-reference and recursion** is the basis for geometry – even though it has partial solutions in arithmetic.

- 1 + 1 = 2 is arithmetic – using the abstract number line
- $\sqrt{1}^2 + \sqrt{1}^2 = \sqrt{2}^2$ is spacial maths **with initial self referential recursive** squaring and rooting
- X + Y = Z is implied spacial algebra
- Log e + i + π = x is spacial algebra with infinity constants in many places

Notice – multivariate analysis of more than 3 variables becomes increasingly complex and difficult to compute using arithmetic. Hence triangulation and distance type spacial maths becomes more usefull.

Hiding infinity and other manipulation

In my previous document I made the point that knowing where you were putting the infinities is important https://humanistman.com/wp-content/uploads/2020/12/Integer-Ratio-Power-Law-Chaos.pdf it also shows the difference in my views and **Brahmagupta** and things like numbers and simple functions like multiplication and summation. The problem **I hypothesize** has to do with tension between the number line as a simple abstraction and the functions which operate. E.g. dividing by zero is a function whereas zero is a position on the abstract number line.

Infinity hides in places in our techniques and symbology but also in plain sight as well.

The Greeks liked the **5 by 5 square** because the diagonal was the square root of 50 (which can also be represented as $5 * \sqrt{2}$) – this looked neat because of its close relationship to 49 which is the result of squaring 7. They then added 1 to make it 50 and $7^2 + 1$ became some kind of special number. Here the infinity is hiding in plain sight as the number 1. This technique of squaring and adding 1 became widely used in many different ways.

This type of hiding infinity also appears in **Ramanujan**'s work https://en.wikipedia.org/wiki/Ramanujan%E2%80%93Sato_series **partial extract Creative Commons Attribution-ShareAlike 3.0**

Ramanujan–Sato series

From Wikipedia, the free encyclopedia

In mathematics, a **Ramanujan–Sato series**[1][2]

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{26390k + 1103}{396^{4k}}$$

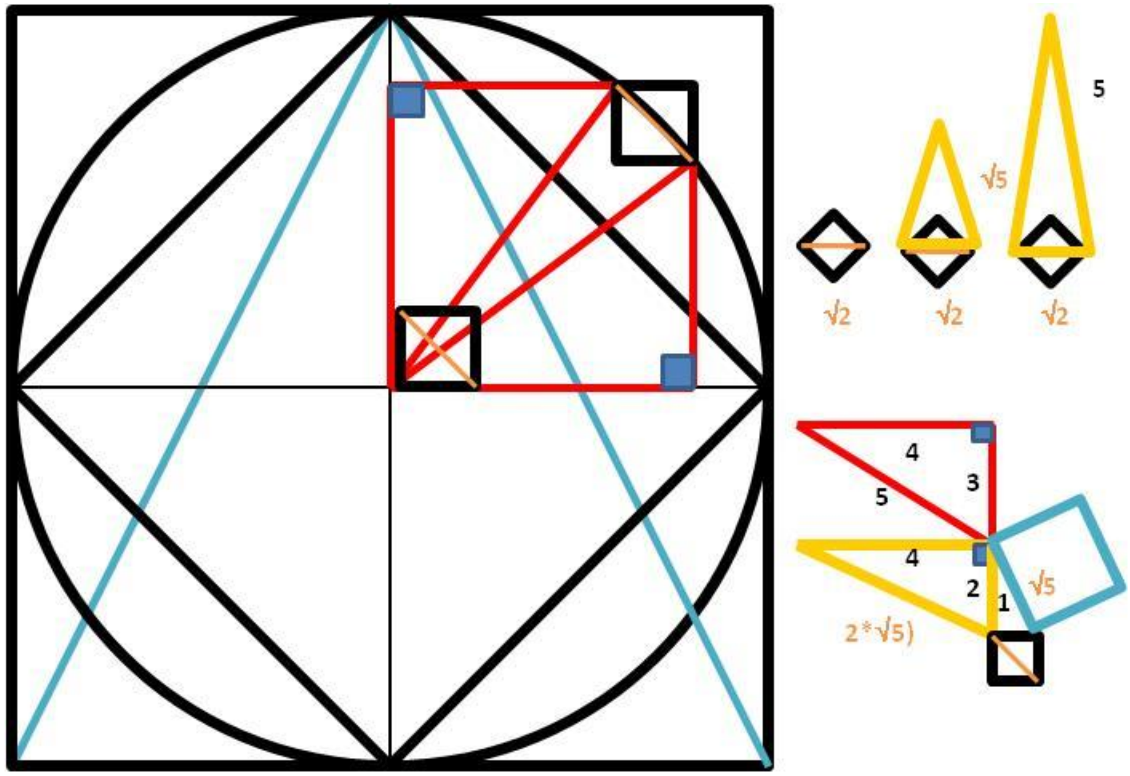
Pi is in plain sight as infinity as is the square root of 2 but what is not so obvious is the 99^2 number – just **one away from a neat number.** A bit like the Greeks hiding a 1 with 49 to make things neat.

The Pell equation can be thought of in terms of the Pythagoras formula. The trick is to use the number 1 to hide infinity in plain sight.

The equation $\mathbf{x^2 - ny^2 = 1}$ is a version of the **Pell equation**.

What is 1? Is it one apple? Is it just a **place holder** for unity to infinity? We know the continuum has to start somewhere why not 1 or zero? at the other end of the continuum is infinity. Another way to think of it in spacial dimensions is to imagine us the observer at the centre of the universe. How far can we see in all directions – let us call that radius r – or as I call it - pi infinity or **one certain unit**.

So the Pell equation is Pythagoras slightly manipulated and changed – but how? Consider the Following Diagram

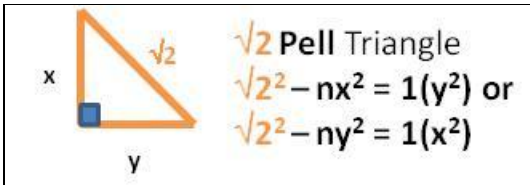


You will note my 10 by 10 square and how everything fits neatly and we get a clear illustration of how the square root of 2 tends to infinity as it goes out towards the circle and that this represents the intersection between two 3,4,5 right angled triangles where their diagonals (5) meet the circle.

(notice too that Plato in the Republic had explored these types of number relationships and extended it to the third dimension – the "volume")

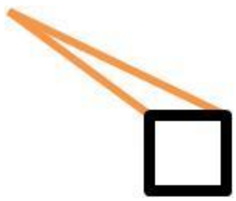
This is where the Pell equation comes in. Instead of looking for prime numbers and patterns starting at zero and moving outwards on the infinity number plane – instead we start out at the edge of the circle and come backwards towards the middle.

So in the Pell equation $\mathbf{x^2 - ny^2 = 1}$ – the 1 here conceptually bares a relationship to the Pythagorean formula – but how? Which side of the formula are we keeping constant – the perpendicular, adjacent or hypothenuse? **Notice** that the use of x and y in the equation above is arbitrary based on conventions – below I actually label the terms according to the triangle.

	As you can see it works for both sides of the triangle – It is trying to find the relationship (the ratio) between the hypotenuse and one other side of the triangle. And this by definition will always tend to the square root of 2

So you can see that the term on the right is really irrelevant for our exploration – what we are looking for is relationships between the hypotenuse and the *other* side. It does not matter which side because if we are exploring we will find all numbers which relate neatly to the hypotenuse being on one side of the triangle or the other. So the Pell equation is really about exploration of the “gap” - the segment - between the 3,4,5 triangles on the large diagram above.

Also notice that this equation is conceptually about “varying the result” – the hypotenuse. So instead of finding the hypotenuse (as one of our biases) we are keeping a non-hypotenuse side fixed at 1 and looking at the other two sides in neat integer relationships.



The other trick worth noticing is that I have effectively – **rebased** the expression in a unit of **√2**. I have hidden it as the x^2 expression – the y and n variables are therefore all in the same base unit – what we are seeing is ratios of sides of triangles. So my equation $\sqrt{2}^2 – ny^2 = 1$ is the same as $x^2 – ny^2 = 1$. The square root of 2 is embedded in the relationship – because any two things squared together related to a third value is a **re-statement** of the right angled triangle – which has the square root of 2 as the basis for one side.

Of course we can add other variables to the triangle – one to multiply the first value as well (the hypotenuse) but what we will see is the square root of 2 infinity always appearing by definition.

Recent Investigations

History of Pell equation and other people’s thoughts on it.

Recent Documents

Brāhmasphuṭasiddhānta :Author(**Brahmagupta**) :Year(628) :Keyword(Individual Development Math)
https://en.wikipedia.org/wiki/Br%C4%81hmasphu%E1%B9%ADasiddh%C4%81nta
https://archive.org/details/Brahmasphutasiddhanta_Vol_1
https://enacademic.com/dic.nsf/enwiki/909484

The Pell Equation :Author(**Edward Everett Whitford**) :Year(1912) :Keyword(Group Development Maths)
https://archive.org/details/pellequation00whitgoog/page/n19/mode/2up

https://www.ebay.com/i/361463552931?chn=ps&mkevt=1&mkcid=28
https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=&cad=rja&uact=8&ved=2ahUKEwi86_yFqvztAhU1_XMBHRLaAdUQFjACegQIARAC&url=https%3A%2F%2Fwww.forgottenbooks.com%2Fen%2Fdownload%2FThePellEquation_10024828.pdf&usg=AOvVaw256iVL1FVbWX1Vo0DaeFNi
*..” Given any **number** which is not a square, there also exists an infinite number of squares such that when multiplied into the given number and unity is added to the product, the result is a square.*
”

*The **problem thus set forth by Fermat** is one of the **most important steps in the history of the Pell equation**. A freer translation of the Latin would read : For every given number which is not a square there exists infinitely many square numbers such that the product of each by the given number, with the addition of 1 ,is a square. Fermat illustrates his problem by a number of examples, one of which is as follows : Given 3 , a non- square number; this number multiplied into the square number 1and 1 being added produces 4 , which is a square. Moreover, the same 3 multiplied into the square 16 with 1 added makes 49, which is a square. And instead of 1 and 16, an infinite number of squares may be found showing the same property ; I demand, however, a general rule, any number being given which is not a square. It is sought, for example, to find a square which when multiplied into 149 109, 433 , etc. , becomes a square when unity is added.”*

***Récréations mathématiques** :Author(**François Édouard Anatole Lucas**) :Year(1891) :Keyword(Group Development Maths) https://archive.org/details/recretionmatedou03lucarich https://hal.archives-ouvertes.fr/hal-01349265 https://sites.google.com/a/books-now.com/en2280/9780265420263-31seslibGEorca85*

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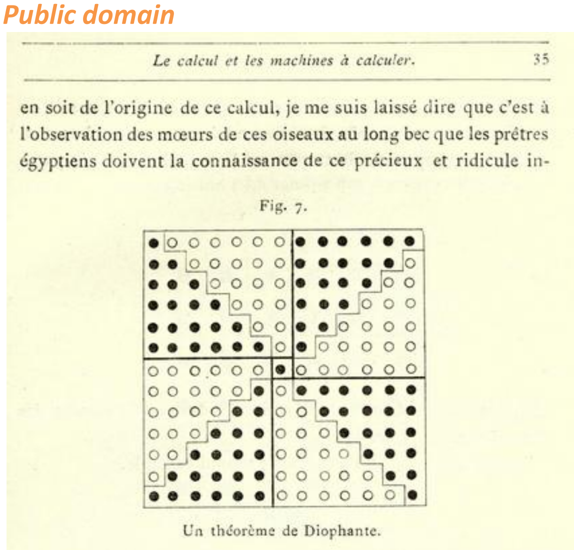
Le calcul et les machines à calculer. 35

en soit de l'origine de ce calcul, je me suis laissé dire que c'est à l'observation des mœurs de ces oiseaux au long bec que les prêtres égyptiens doivent la connaissance de ce précieux et ridicule in-

Fig. 7.

Un théorème de Diophante.

Recent People

Brahmagupta :Year(598-670) :Keyword(Math) https://mathshistory.st-andrews.ac.uk/Biographies/Brahmagupta/
https://www.storyofmathematics.com/indian_brahmagupta.html
http://www.educ.fc.ul.pt/icm/icm2003/icm14/Brahmagupta.htm

Varahamihira :Year(505-587) :Keyword(Math, Astronomy) https://mathshistory.st-andrews.ac.uk/Biographies/Varahamihira/

Edward Everett Whitford :Year(1865) :Keyword(Math)
https://archive.org/details/pellequation00whitgoog/page/n19/mode/2up
https://www.ams.org/journals/bull/1915-21-07/
https://quod.lib.umich.edu/u/umhistmath/ABV2773.0001.001?rgn=main;view=fulltext

François Édouard Anatole Lucas :Year(1842-1891) :Keyword(Math)
https://en.wikipedia.org/wiki/%C3%89douard_Lucas https://mathshistory.st-andrews.ac.uk/Biographies/Lucas / http://edouardlucas.free.fr/oeuvres/recreations_math_01_lucas.pdf

References

1. ***Wikipedia*** *https://en.wikipedia.org/wiki/Main_Page* ***Larry Sanger*** *and* ***Jimmy Wales*** *https://en.wikipedia.org/wiki/Founder_of_Wikipedia* ***Creative Commons Attribution-ShareAlike 3.0***
2. ***Many Universities, government , museum, library and public websites***
3. ***Internet Archive*** *- Internet Archive Founder,* ***Brewster Kahle*** *- https://archive.org/*
4. ***Project Gutenberg*** *-* ***Michael Hart****, founder of Project Gutenberg, invented eBooks in 1971 and his memory continues to inspire the creation of eBooks and related content today. https://www.gutenberg.org/* **public domain**
5. ***MacTutor*** *https://mathshistory.st-andrews.ac.uk/ "MacTutor is created and maintained by* ***Edmund Robertson****, and* ***John O'Connor****, of the School of Mathematics and Statistics at the* ***University of St Andrews****, and is hosted by the University. Their contributions to the history of mathematics have been recognised by the Comenius medal of the Hungarian Comenius Society in 2012 and the Hirst Prize of the London Mathematical Society in 2015."* **Copyright - Creative Commons Attribution-ShareAlike 4.0 International License.**
6. ***Mathematical Association Of America https://www.maa.org/***
7. ***American Mathematical Association https://www.ams.org/home/page***
8. ***Cornell University - arXiv.org*** *arXiv is a free distribution service and an open-access archive for 1,817,857 scholarly articles in the fields of physics, mathematics, computer science, quantitative biology, quantitative finance, statistics, electrical engineering and systems science, and economics. Materials on this site are not peer-reviewed by arXiv.*
9. ***Pell*** *https://mathshistory.st-andrews.ac.uk/HistTopics/Pell/*
10. ***Pell's equation*** *https://en.wikipedia.org/wiki/Pell%27s_equation* ***Creative Commons Attribution-ShareAlike 3.0*** *.." Pell's equation, also called the Pell–Fermat equation, is any Diophantine equation of the form x 2 − n y 2 = 1 {\displaystyle x^{2}-ny^{2}=1} x^2-ny^2=1 where n is a given positive nonsquare integer and integer solutions are sought for x and y. In Cartesian coordinates, the equation has the form of a hyperbola; solutions occur wherever the curve passes through a point whose x and y coordinates are both integers, such as the trivial solution with x = 1 and y = 0. Joseph Louis Lagrange proved that, as long as n is not a perfect square, Pell's equation has infinitely many distinct integer solutions. These solutions*

*may be used to **accurately approximate the square root of n by rational numbers** of the form x/y."*

11. ***William Brouncker**, 2nd Viscount Brouncker* *https://en.wikipedia.org/wiki/William_Brouncker,_2nd_Viscount_Brouncker*
12. ***Ed Barbeau** http://www.math.toronto.edu/barbeau/*
13. *The Mathematics Enthusiast The Mathematics Enthusiast Volume 8 Number 1 Numbers 1 & 2 Article 16 1-2011 Gifted Students and Advanced Mathematics Gifted Students and Advanced Mathematics **Edward J. Barbeau** https://scholarworks.umt.edu/cgi/viewcontent.cgi?article=1218&context=tme Barbeau, Edward J. (2011) "Gifted Students and Advanced Mathematics," The Mathematics Enthusiast: Vol. 8 : No. 1 , Article 16. Available at: https://scholarworks.umt.edu/tme/vol8/iss1/16 This Article is brought to you for free and open access by ScholarWorks at University of Montana. It has been accepted for inclusion in The Mathematics Enthusiast by an authorized editor of ScholarWorks at University of Montana. For more information, please contact scholarworks@mso.umt.edu.- **copyright - partial extract – fair use ..** "§4. Conclusion. In dealing with gifted students, the guiding principle should be to **broaden the experience** of the students at each level, and not to proceed to more advanced work unless it is carefully prepared for. Advanced mathematics involves **more abstraction and generality**, and so is inclined to increase the intuitive distance between the student and the mathematics, unless the intuition itself is enriched. There is a trade-off between the intelligibility of particular situations presented at a lower level and their capacity for inclusion in a broader sphere at a higher level. To appreciate the power and elegance of higher mathematics, and to exploit it judiciously, students need time and experience to develop comfort and facility with sophisticated matter."*
14. ***The Pell Equation** – **Edward Everett Whitford** - 1912 https://archive.org/details/pellequation00whitgoog **public domain** .."It will not be far out of the way to say that the first approximations to √2 appeared both in India and in Greece about four hundred years before Christ. The younger Pythagoreans (before 410 BC.) recognized and proved the incommensurability of the diagonal and side of a square and set forth certain approximations. The Sulva-sutras in India, which contained approximations to √2, are not later than the fourth or fifth century before Christ."*

15.

10 THE PELL EQUATION

taken as a line 5 units in length and upon it a square was constructed. This square contained, accordingly, 25 square units. The diagonal of this square was $\sqrt{2 \cdot 25}$ linear units. This diagonal was called by the Pythagoreans the ἄρρητος διάμετρος of the number 5. In conjunction with this $\sqrt{50-1}$ was called the ῥητὴ διάμετρος which equaled 7 linear units, i.e., the rational number whose square differs from the square of the irrational diagonal by an integral minimum. And 7, 5, is a solution of

$$x^2 - 2y^2 = -1.$$

That the ῥητὴ διάμετρος had a general significance to the Pythagoreans has become evident since the publication of Kroll's[1] text of Proclus's commentary on Plato. It is based on an application of Euclid II, 10,[2] which was common property before Plato and was known to the Pythagoreans as early as the middle of the fifth century before Christ. If C (Fig. 1) is the mid-point of the base

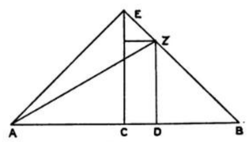

Fig. 1.

of an isosceles right triangle AEB, and D any point on AB, then

$$\overline{AD}^2 + \overline{DB}^2 = 2\overline{AC}^2 + 2\overline{CD}^2.$$

For if Z is the point where the perpendicular at D meets

[1] Proclus, "In Platonis rempublicam commentarii," editor Kroll, vol. II, c. 27, Leipzig, 1901.

[2] F. Hultsch, loc. cit. H. G. Zeuthen, "Geschichte der Mathematik im Altertum und Mittelalter," p. 59, Copenhagen, 1896. Also the French edition, H. G. Zeuthen, "Histoire des mathématiques dans l'antiquité et le moyen age," J. Mascart, ed., p. 47, Paris, 1902. M. Cantor, op. cit., vol. I, p. 249, 2d ed.

THE PELL EQUATION 11

EB,

$$DB = DZ$$

$$\overline{AD}^2 + \overline{DZ}^2 = \overline{AE}^2 + \overline{EZ}^2 = 2\overline{AC}^2 + 2\overline{CD}^2.$$

If

$$CD = q \quad \text{and} \quad BD = p,$$

$$AD = 2q + p = p_1, \quad AC = q + p = q_1,$$

then

$$2q_1^2 - p_1^2 = -(2q^2 - p^2).$$

This enables us to derive from one integral solution, p, q, of one of the two equations

$$x^2 - 2y^2 = \pm 1,$$

a solution, p_1, q_1, for the other, always in larger integers. We might start from 1, 1, or from the one with which the Pythagoreans were already familiar, 7, 5.

As Proclus[1] pointed out, the Pythagoreans proceeded as follows:[2] On AB (Fig. 2) construct a square and draw its diagonal BE. On the prolongation of AB lay off

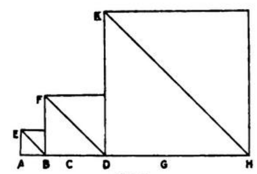

Fig. 2.

$BC = AB$ and $CD = BE$. Then according to the Pythagorean theorem

$$\overline{CD}^2 = 2\overline{AB}^2,$$

and by the theorem just referred to

$$\overline{AD}^2 + \overline{CD}^2 = 2\overline{AB}^2 + 2\overline{BD}^2.$$

[1] Proclus, loc. cit.

[2] Hultsch, loc. cit.

16. *Digital Mathematics Library © 2003 -- 2013 Ulf Rehmann Fakultät für Mathematik, Universität Bielefeld, Germany https://www.math.uni-bielefeld.de/~rehmann/DML/dml_links_title_P.html*
17. ***Surds*** *https://amsi.org.au/teacher_modules/Surds.html* (I generally support the quality of this work – it highlights **important current rules** but ignores deeper issues)
18. ***Theon of Smyrna*** *https://mathshistory.st-andrews.ac.uk/Biographies/Theon_of_Smyrna/*
19. ***Thomas Fantet de Lagny*** *https://en.wikipedia.org/wiki/Thomas_Fantet_de_Lagny , https://mathshistory.st-andrews.ac.uk/Biographies/Lagny/ …" Lagny constructed trigonometric tables and used binary arithmetic in his text Trigonométrie française ou reformée published in Rochefort in 1703. In 1733 he examined the continued fraction expansion of the quotient of two integers and, as an example, considered adjacent Fibonacci numbers as the worst case expansion for the Euclidean algorithm in his paper Analyse générale ou Méthodes nouvelles pour résoudre les problèmes de tous les genres et de tous les degrés à l'infini."*
20. ***Hieronymus Georg Zeuthen*** *https://en.wikipedia.org/wiki/Hieronymus_Georg_Zeuthen*
21. ***J.B.H. Heilermann,*** *De transformatione serierum in fractiones continuas, Diss. Münster, 1845. Google Scholar https://www.sciencedirect.com/science/article/pii/0377042789903324 , https://www.researchgate.net/publication/226338103_History_of_Continued_Fractions_and_Pade_Approximants*
22. *Proc. Natl. Acad. Sci. USA Vol. 93, pp. 15004–15008, December 1996 Mathematics New infinite families of exact sums of squares formulas, Jacobi elliptic functions, and Ramanujan's tau function(Jacobi continued fractions Hankel or Turánian determinants Fourier series*

Lambert series Schur functions) ***STEPHEN C. MILNE*** *Department of Mathematics, Ohio State University, 231 West 18th Avenue, Columbus, OH 43210 Communicated by Walter Feit, Yale University, New Haven, CT, October 22, 1996 (received for review June 24, 1996* *https://www.pnas.org/content/pnas/93/26/15004.full.pdf*

23. *Continued fractions-* ***Henri Eugène Padé*** *https://mathshistory.st-andrews.ac.uk/Biographies/Pade/ , https://en.wikipedia.org/wiki/Pad%C3%A9_approximant , https://pantheon.world/profile/person/Henri_Pad%C3%A9/*
24. ***Nicolas Chuquet*** *https://en.wikipedia.org/wiki/Nicolas_Chuquet , https://mathshistory.st-andrews.ac.uk/Biographies/Chuquet/ ..." Nicolas Chuquet was a French mathematician who wrote the earliest French algebra book."*
25. ***Juan Ortega*** *https://en.wikipedia.org/wiki/Juan_de_Ortega_(mathematician) , https://mathshistory.st-andrews.ac.uk/Biographies/Ortega/ , http://galileo.rice.edu/Catalog/NewFiles/ortega.html*
26. ***The Pell Equation - Edward Everett Whitford*** *(page 47) letter from Fermat (Oevres de Fermat volume II page 333) "Fermat first proposed the general problem in a letter written to Frenicle in February, 1657 "https://archive.org/details/pellequation00whitgoog/page/n57/mode/2up "There is scarcely any one who states purely arithmetical questions,* ***scarcely any who understands them****. Is this not because arithmetic has been treated up to this time geometrically rather than arithmetically? This certainly is indicated by many works ancient and modem. Diophantus himself also indicates this. But he has freed himself from geometry a little more than others have, in that he limits his analysis to rational numbers only; nevertheless the Zetcica of Vieta, in which the methods of Diophantus are extended to continuous magnitude and therefore to geometry, witness the* ***insufficient separation of arithmetic from geometry****."*
27. ***George Ballard Mathews*** *https://en.wikipedia.org/wiki/George_Ballard_Mathews , Theory Of Numbers https://archive.org/details/theoryofnumbers00math*
28. *FERMAT'S DILEMMA: WHY DID HE KEEP MUM ONINFINITESIMALS? AND THE EUROPEANTHEOLOGICAL CONTEXTJACQUES BAIR, MIKHAIL G. KATZ, AND DAVID SHERRY –* ***copyright assumed – fair use – reference only*** *https://arxiv.org/pdf/1801.00427.pdf , https://arxiv.org/abs/1801.00427v1*
29. ***Weierstrass function*** *https://en.wikipedia.org/wiki/Weierstrass_function*
30. ***Biodiversity Heritage Library*** *- Oeuvres de Fermat* ***copyright assumed – fair use – reference only*** *https://www.biodiversitylibrary.org/bibliography/22243#/summary*
31. ***Clark University*** *- Euclid's Elements Book II Proposition 10* ***copyright assumed – fair use – reference only*** *https://mathcs.clarku.edu/~djoyce/elements/bookII/propII10.html*
32. *TY - BOOK AU -* ***Baloglou, George*** *AU -* ***Thomaidis, Yannis*** *PY - 2020/11/14 SP - T1 -* ***From Plato's Rational Diameter to Proclus' Elegant Theorem*** *ER -* ***copyright assumed – fair use – reference only*** *https://www.researchgate.net/publication/345971352_From_Plato%27s_Rational_Diameter_to_Proclus%27_Elegant_Theorem*

33. ***Plato's Number*** *https://en.wikipedia.org/wiki/Plato%27s_number*
34. ***The Republic by Plato*** *https://www.gutenberg.org/files/1497/1497-h/1497-h.htm* **public domain** *..." After this manner:—A city which is thus constituted can hardly be shaken; but, seeing that everything which has a beginning has also an end, even a constitution such as yours will not last for ever, but will in time be dissolved. And this is the dissolution:—In plants that grow in the earth, as well as in animals that move on the earth's surface, fertility and sterility of soul and body occur when the circumferences of the circles of each are completed, which in short-lived existences pass over a short space, and in long-lived ones over a long space. But to the knowledge of human fecundity and sterility all the wisdom and education of your rulers will not attain; the* ***laws which regulate them will not be discovered by an intelligence which is alloyed with sense, but will escape them****, and they will bring children into the world when they ought not. Now that which is of divine birth has a period which is contained in a* ***perfect number*** *(i.e. a* ***cyclical number, such as 6, which is equal to the sum of its divisors 1, 2, 3,*** *so that when the circle or time represented by 6 is completed, the lesser times or rotations represented by* ***1, 2, 3 are also completed****.), but the period of human birth is comprehended in a number in which first increments by involution and evolution (or squared and cubed) obtaining three intervals and four terms of like and unlike, waxing and waning numbers, make all the terms commensurable and agreeable to one another. (Probably the numbers 3, 4, 5, 6 of which the three first = the sides of the Pythagorean triangle. The terms will then be 3 cubed, 4 cubed, 5 cubed, which together = 6 cubed = 216.) The base of these (3) with a third added (4) when combined with five (20) and raised to the third power furnishes two harmonies; the first a square which is a hundred times as great (400 = 4 x 100) (Or the first a square which is 100 x 100 = 10,000. The whole number will then be 17,500 = a square of 100, and an oblong of 100 by 75.), and the other a figure having one side equal to the former, but oblong, consisting of a hundred numbers squared upon rational diameters of a square (i.e. omitting fractions), the side of which is five (****7 x 7 = 49*** *x 100 = 4900), each of them being less by one (than the perfect square which includes the fractions, sc. 50) or less by (Or, 'consisting of two numbers squared upon irrational diameters,' etc. = 100. For other explanations of the passage see Introduction.) two perfect squares of irrational diameters (of a square the side of which is five = 50 + 50 = 100); and a hundred cubes of three (****27 x 100 = 2700 + 4900 + 400*** *=* ***8000****). Now this number represents a geometrical figure which has control over the good and evil of births. For when your guardians are ignorant of the law of births, and unite bride and bridegroom out of season, the children will not be goodly or fortunate. And though only the best of them will be appointed by their predecessors,* ***still they will be unworthy to hold their fathers' places****, and when they* ***come into power as guardians****, they will soon be* ***found to fail*** *in taking care of us, the Muses, first by under-valuing music; which neglect will soon extend to gymnastic; and hence the young men of your State will be less cultivated. In the* ***succeeding generation rulers will be appointed who have*** ***lost the guardian power of testing the metal*** *of your different races, which, like Hesiod's, are of gold and silver and brass and iron. And so iron will be mingled with silver, and brass with gold, and hence there will arise dissimilarity and inequality and irregularity, which always and in all places are causes of hatred and war.* ***This***

the Muses affirm to be the stock from which discord has sprung, wherever arising; and this is their answer to us."…

35. ***Arthur Eddington*** *https://en.wikipedia.org/wiki/Arthur_Eddington*
36. ***Ehrenfest paradox*** *https://en.wikipedia.org/wiki/Ehrenfest_paradox*
37. ***The Theory of Simply Periodic Numerical Functions Edouard Lucas*** *First published in the American Journal of Mathematics, Vol. 1 (1878), pp. 184-240 and 289-321. Translated from the French by Sidney Kravitz. Translation edited by Douglas Lind. Published 1969 by the Fibonacci Association. https://www.fq.math.ca/simply-periodic.html* ***You may download the entire volume (size: 4.6Mb) for free.***
38. ***The Puzzle Museum*** *https://www.puzzlemuseum.com/*